Lecture V: Bogoliubov Theory of weakly interacting Bose gas

References:

e.g. Abrikosov, Gor'kov and Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics*, Dover

The theory of superfluidity is better postponed to the field integral discussion next term. However, a microscopic theory of the weakly interacting Bose gas can be developed within a second quantised picture. Here we discuss the effect of weak interactions on the nature of the Bose-Einstein condensed state and the spectrum of low-lying excitations around the ground state.

▷ Weakly interacting Bose gas

Consider N bosons confined to a volume L^d, density $n = N/L^d$,

In non-interacting system, at $T = 0$, all bosons condense
into the lowest energy single-particle state $|\text{g.s.}\rangle = (a_0^\dagger)^N |\Omega\rangle$

How is ground state influenced by weak interaction?

$$\hat{H} = \sum_{\mathbf{k}} \frac{\hbar^2 \mathbf{k}^2}{2m} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \overbrace{\frac{1}{2}\int d\mathbf{x} \int d\mathbf{x}'\, a^\dagger(\mathbf{x}) a^\dagger(\mathbf{x}') V(\mathbf{x}-\mathbf{x}') a(\mathbf{x}') a(\mathbf{x})}^{\hat{H}_I}$$

$V(\mathbf{x}-\mathbf{x}')$ — pairwise particle interaction

In Fourier basis:

$$\hat{H}_I = \frac{1}{2L^d} \sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} V(\mathbf{q})\, a_{\mathbf{k}+\mathbf{q}}^\dagger a_{\mathbf{k}'}^\dagger a_{\mathbf{k}} a_{\mathbf{k}'+\mathbf{q}}$$

$a_{\mathbf{k}} = L^{-d/2} \int d\mathbf{x}\, e^{i\mathbf{k}\cdot\mathbf{x}} a(\mathbf{x})$, and $V(\mathbf{q}) = \int d\mathbf{x}\, e^{-i\mathbf{q}\cdot\mathbf{x}} V(\mathbf{x})$.

Suppose interaction weak — in condensed phase, we may assume that the lowest-lying
single particle state is still macroscopically occupied, i.e. $N_0/N = O(1)$

Since $\hat{N}_0 = a_{k=0}^\dagger a_{k=0} = O(N) \gg 1$, the commutator

$$a_0 a_0^\dagger - a_0^\dagger a_0 = 1$$

is small compared to a_0 and $a_0^\dagger$ themselves, i.e. to a good approximation,
field operators can be replaced by an ordinary c-number $\sqrt{N_0}$

Then, taking $V(\mathbf{q}) = V$ const., i.e. a contact interaction
momentum conservation identies the contributions:

$$\hat{H}_I = \frac{V}{2L^d} N_0^2 + \frac{V}{L^d} N_0 \sum_{\mathbf{k}\neq 0} \left[a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + a_{-\mathbf{k}}^\dagger a_{-\mathbf{k}} + \frac{1}{2}\left(a_{-\mathbf{k}} a_{\mathbf{k}} + a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger \right) \right] + O(N_0^0)$$

cf. AF Hamiltonian in spin-wave approximation

▷ Physical interpretation of components:

- $Va^\dagger_{\mathbf{k}}a_{\mathbf{k}}$ represents the 'Hartree-Fock energy' of excited particles interacting with condensate — *note that the contact nature of the interaction disguises the presence of the direct and exchange contributions*
- $V(a_{-\mathbf{k}}a_{\mathbf{k}} + a^\dagger_{\mathbf{k}}a^\dagger_{-\mathbf{k}})$ represents creation or annihilation of excited particles from the condensate; *in the approximation, total no. of particles is not conserved*

Final part of approximation involves noting that $N = N_0 + \sum_{\mathbf{k}\neq 0} a^\dagger_{\mathbf{k}}a_{\mathbf{k}}$,
and using this identity to trade $\hat{N}_0$

$$\hat{H} = \frac{VnN}{2} + \sum_{\mathbf{k}\neq 0}\left[\left(\epsilon^0_{\mathbf{k}} + Vn\right)\, a^\dagger_{\mathbf{k}}a_{\mathbf{k}} + \frac{Vn}{2}\left(a_{-\mathbf{k}}a_{\mathbf{k}} + a^\dagger_{\mathbf{k}}a^\dagger_{-\mathbf{k}}\right)\right]$$

$n = N/V$ — total density and $\epsilon^0_{\mathbf{k}} = \frac{\hbar^2\mathbf{k}^2}{2m}$

As with spin wave theory, Hamiltonian diagonalised by Bogoluibov transformation:

$$\begin{pmatrix} a_{\mathbf{k}} \\ a^\dagger_{-\mathbf{k}} \end{pmatrix} = \begin{pmatrix} \cosh\theta_{\mathbf{k}} & -\sinh\theta_k \\ -\sinh\theta_{\mathbf{k}} & \cosh\theta_{\mathbf{k}} \end{pmatrix}\begin{pmatrix} \alpha_{\mathbf{k}} \\ \alpha^\dagger_{-\mathbf{k}} \end{pmatrix}$$

Left as exercise to show that, when

$$\cosh^2\theta_{\mathbf{k}} = \frac{1}{2}\left(\frac{\epsilon^0_{\mathbf{k}} + Vn}{\epsilon_{\mathbf{k}}} + 1\right), \qquad \sinh^2\theta_{\mathbf{k}} = \frac{1}{2}\left(\frac{\epsilon^0_{\mathbf{k}} + Vn}{\epsilon_{\mathbf{k}}} - 1\right)$$

$$\hat{H} = \frac{VnN}{2} - \frac{1}{2}\sum_{\mathbf{k}\neq 0}(\epsilon^0_{\mathbf{k}} + nV - \epsilon_{\mathbf{k}}) + \sum_{\mathbf{k}\neq 0}\epsilon_{\mathbf{k}}\alpha^\dagger_{\mathbf{k}}\alpha_{\mathbf{k}}, \qquad \epsilon_{\mathbf{k}} = \left[\left(\epsilon^0_{\mathbf{k}} + Vn\right)^2 - (Vn)^2\right]^{1/2}$$

Spectrum of low-energy excitations scales linearly as $\epsilon(\mathbf{k}) \simeq \hbar c|\mathbf{k}|$
where the velocity $c = (Vn/m)^{1/2}$

High-energy spectrum is free particle-like, i.e. when $k \sim k_0 = mc/\hbar$,
effect of interaction is to displace particles from condensate even at $T = 0$

▷ Ground state wavefunction: defined by condition $\alpha_{\mathbf{k}}|\text{g.s.}\rangle$

Now Bogoluibov transformation can be written as $\alpha_{\mathbf{k}} = Ua_{\mathbf{k}}U^{-1}$ where (exercise)

$$U = \exp\left[\sum_{\mathbf{k}\neq 0}\frac{\theta_{\mathbf{k}}}{2}(a^\dagger_{\mathbf{k}}a^\dagger_{-\mathbf{k}} - a_{\mathbf{k}}a_{-\mathbf{k}})\right]$$

Since g.s. of ideal Bose gas given by $a_{\mathbf{k}}|\Phi_0\rangle = 0$ (all particles in $\mathbf{k} = 0$ state),
new ground state is $|\Phi_V\rangle = U|\Phi_0\rangle$ — proof:

$$0 = a_{\mathbf{k}=0}|\Phi_0\rangle = U^{-1}\overbrace{Ua_{\mathbf{k}}U^{-1}}^{\alpha_{\mathbf{k}}}U|\Phi_0\rangle$$

▷ Depletion of condensate by interaction

$$\frac{N-N_0}{N} = \frac{1}{N}\sum_{\mathbf{k}\neq 0}\langle \text{g.s.}|a^\dagger_{\mathbf{k}}a_{\mathbf{k}}|\text{g.s.}\rangle = \frac{1}{N}\sum_{\mathbf{k}\neq 0}\sinh^2\theta_{\mathbf{k}} = \frac{1}{n}\int\frac{d\mathbf{k}}{(2\pi)^3}\sinh^2\theta_{\mathbf{k}} = \frac{1}{3\pi^2 n}k_0^3$$

i.e. ca. one particle per 'coherence length' $\xi \sim 1/k_0$

Recast using scattering length of contact interaction $V = 4\pi\hbar^2 a/m$,

$$\frac{N-N_0}{N} = \frac{8}{3\sqrt{\pi}}(na^3)^{1/2}$$

▷ Ground state energy

$$E_0 = \frac{VnN}{2} - \frac{1}{2}\sum_{\mathbf{k}\neq 0}(\epsilon^0_{\mathbf{k}} + nV - \epsilon_{\mathbf{k}} - \frac{(nV)^2}{2\epsilon^0_{\mathbf{k}}})$$

(where extra term controls unphysical divergence UV divergence
required for contact potential approximation)

$$\frac{E_0}{L^d} = \frac{n^2 V}{2}\left[1 + \frac{128}{15\sqrt{\pi}}(na^3)^{1/2}\right]$$

▷ Experiment

When cooled to $T \sim 2K$, liquid ^{4}He undergoes transition to superfluid state

In Helium, steric interactions are strong — at higher energy scales
an important second branch of excitations known as rotons appear

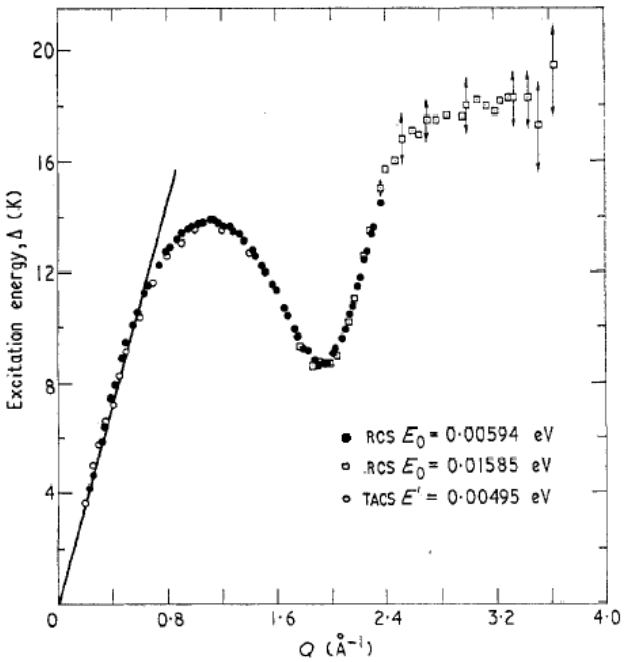
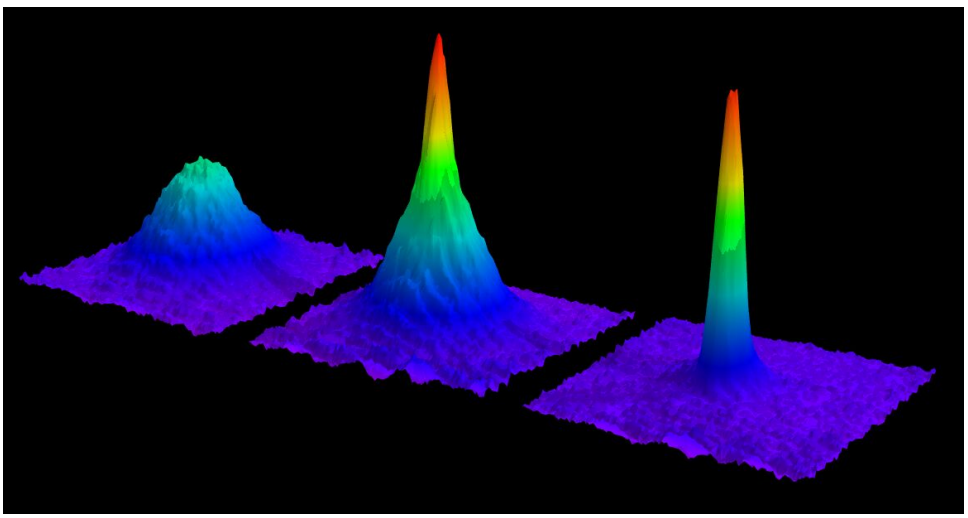
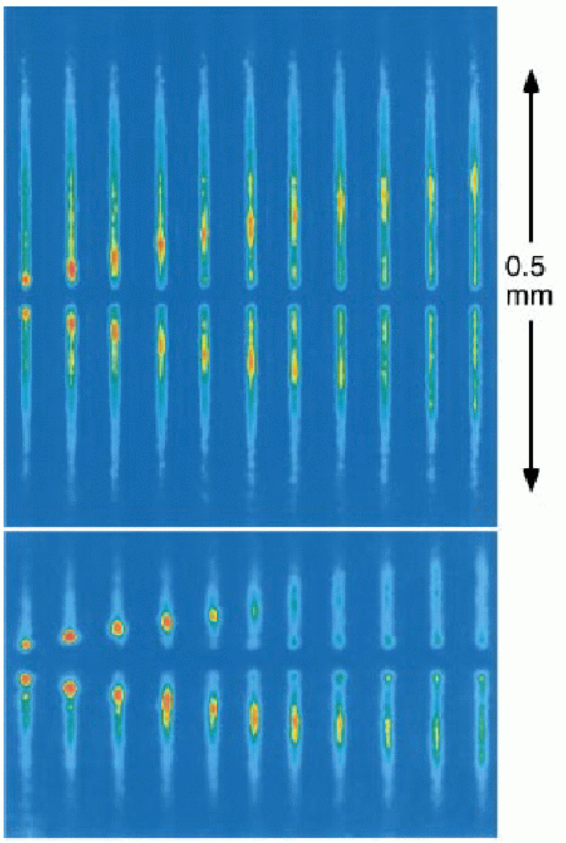


Figure 9. The dispersion curve for He II for the elementary excitations at $T = 1{\cdot}1$ K and $P \simeq 0$. Data obtained by rotating crystal spectrometer (RCS) and triple axis crystal spectrometer (TACS) as indicated. From Cowley and Woods (1971).





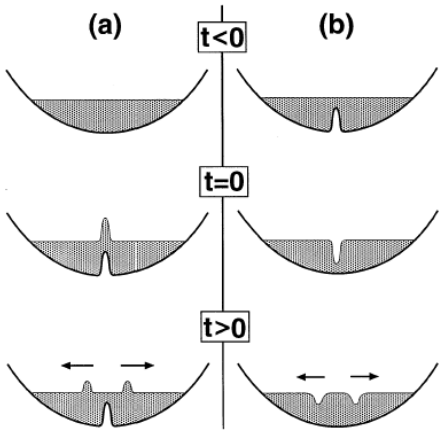


FIG. 1. Excitations of wave packets in a Bose-Einstein condensate. A condensate is confined in the potential of a magnetic trap. At time $t = 0$, a focused, blue-detuned laser beam is suddenly switched on (a) or off (b) and, by the optical dipole force, creates, respectively, two positive or negative pertubations in density which propagate at the speed of sound.

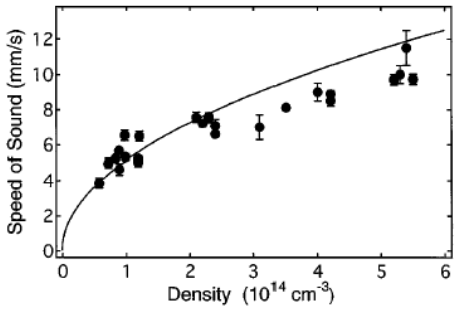


FIG. 4. Speed of sound versus condensate peak density. The solid line is the speed of sound [Eq. (2)] using the maximum cloud density [Eq. (4)] with no adjustable parameter. The error bars show only the statistical error.

M. R. Andrews, *et al.* Phys. Rev. Lett. **79**, 553 (1997).