Lectures VI: Correlated Quantum Spins in One Dimension[3]

▷ References:

Original papers:

- H. Bethe, Z. Phys. **71**, 205 (1931), for an exact diagonalisation of the $S = 1/2$ Heisenberg chain; method not useful for correlation functions.
- A. Luther, Phys. Rev. B **14**, 2153 (1976); A. Luther and I.Peschel, Phys. Rev. B **12**, 3908 (1975), for mapping of $S = 1/2$ Heisenberg chain to free massless bosons, and correlation functions.

Books and Reviews:

- A. O. Gogolin, A. A. Nersesyan, and A. M. Tsvelik, *Bosonization and Strongly Correlated Systems* (Cambridge University Press, 1998).
- H. J. Schulz, G. Cuniberti, and P. Pieri, *Fermi Liquids and Luttinger Liquids*, `cond-mat/9807366`.

▷ Notable features

(a) The $S = 1/2$ antiferromagnetic chain never orders, however at $T = 0$, it is critical, i.e., correlation functions decay algebraically.

(b) The $S = 1$ chain develops a spin gap $\Rightarrow$ uncorrelated over long distances.

▷ The Heisenberg $S = \frac{1}{2}$ chain

$$H = J \sum_{\langle ij \rangle} \left[\frac{1}{2} \left(S_i^+ S_j^- + S_i^- S_j^+ \right) + \Delta S_i^z S_j^z \right] \tag{1}$$

- At any given site i, the spin operators S_i^-, S_i^+ anticommute,

$$\{S_i^+, S_i^-\} = 1,$$

prompting us to make an analogy with fermions,

$$S_i^+ = \psi_i^\dagger, \; S_i^- = \psi_i, \; S_i^z = \psi_i^\dagger \psi_i - \frac{1}{2}. \tag{2}$$

Unfortunately, the analogy stops here for spins at different sites *commute* whereas fermions at different sites *anticommute*, e.g.,

$$\{\psi_i, \psi_j^\dagger\} = 0, \; [S_i^+, S_j^-] = 0 \qquad i \neq j.$$

[3] Lecture notes prepared by Vikram Tripathi

Another angle: The spin operators S_i^-, S_j^+ behave as annihilation and creation operators for bosons at different sites ($i \neq j$). The presence or absence of a boson at a site can be interpreted as spin-up and spin-down respectively. Of course, you are not allowed to put two bosons at any site because you can only have either spin up or spin down, and not both. Thus the bosons have a hard-core character. And hard-core bosons prevent double-occupation at any site which is a property of fermions also since two identical fermions will never occupy the same site.

What should be done? We need to modify the analogy in Eq.(2) appropriately so that an extra negative sign is introduced when commuting spins at different sites. The transformation was obtained by P. Jordan and E. Wigner, Z. Phys. **47**, 631 (1928).

▷ (1928) The Jordan-Wigner transformation in 1D

Write

$$\begin{aligned} S_i^+ &= \psi_i^\dagger \exp\left[i\pi \sum_{j<i} \hat{n}_j\right] \\ S_i^- &= \exp\left[-i\pi \sum_{j<i} \hat{n}_j\right] \psi_i, \qquad \hat{n}_j = \psi_j^\dagger \psi_j. \end{aligned}$$

The Jordan-Wigner factors $\exp\left[\pm i\pi \sum_{j<i} \hat{n}_j\right]$ introduce just that extra negative sign because of relations such as

$$\exp[i\pi n_j]\psi_j = \psi_j, \qquad \text{and} \qquad \psi_j \exp[i\pi n_j] = \psi_j e^{i\pi} = -\psi_j.$$

• *Physical meaning of the transformation*: One way to interpret the transformations is to consider fermions to which a single flux quantum is attached. When two such 'composite-fermions' are exchanged, their relative displacement is rotated by π. This is what the Jordan-Wigner phase factor counts.

Consider a state with two particles labelled A and B at sites i and j. Let B be to the right of A and let there be n particles in between. Exchanging A with B involves exchanging A with each of the n particles to its right, then an exchange with B, and finally and exchange of B with n particles to its left. That results in accumulation of an extra factor of $(-1)^n \times (-1) \times (-1)^n = -1$.

▷ (1989–1993) The Jordan-Wigner transformation in 2D and 3D

Defining moments: E. Fradkin, Phys. Rev. Lett. **63**, 322 (1989); D. Elizer and G. W. Semenoff, Phys. Lett. B. **286**, 118 (1992); L. Huerta and J. Zanelli, Phys. Rev. Lett. **71**, 3622 (1993).

In 2D, the transformation (due to E. Fradkin) is

$$S_i^+ = \psi_i^\dagger \exp\left[i \sum_{j\neq i} \arg(\mathbf{j} - \mathbf{i})\, n_j\right].$$

The transformation in 3D was obtained by L. Huerta and J. Zanelli: Phys. Rev. Lett. **71**, 3622 (1993).

- The $S = 1/2$ Heisenberg chain in 1D can now be recast as

$$H = J \sum_{\langle i=1 \rangle}^{N} \left[\frac{1}{2} \left(\psi_i^\dagger \psi_{i+1} + \psi_{i+1}^\dagger \psi_i \right) + \Delta \left(\psi_i^\dagger \psi_i - \frac{1}{2} \right) \left(\psi_{i+1}^\dagger \psi_{i+1} - \frac{1}{2} \right) \right] \tag{3}$$

Observe that $\Delta = 0$ corresponds to non-interacting fermions. The 'kinetic' term may be diagonalised going over to the momentum representation,

$$\psi_j^\dagger = \frac{1}{\sqrt{N}} \sum_k e^{ikj}\, \psi^\dagger(k);$$

the Hamiltonian takes the form

$$H = J \sum_k (\cos(k) - \Delta) \psi^\dagger(k) \psi(k) + \frac{J\Delta}{2N} \sum_{p,q,k} e^{iq} \psi^\dagger(p+q) \psi^\dagger(k-q) \psi(k) \psi(p) + \frac{JN\Delta}{4}.$$

It is remarkable that the excitations of a chain of Heisenberg spins coupled antiferromagnetically in the XY plane are the same as non-interacting electrons! This feature is peculiar to 1D because of the close similarity between hard-core bosons and fermions: they are related to each other through a counting factor, the Jordan-Wigner phase, and counting doesn't cost energy. In higher dimensions, the Jordan-Wigner phase contributes to the low energy excitations, hence the excitation spectrum is no longer that of non-interacting fermions.

(a) Consider $\Delta = 0$: The XY chain

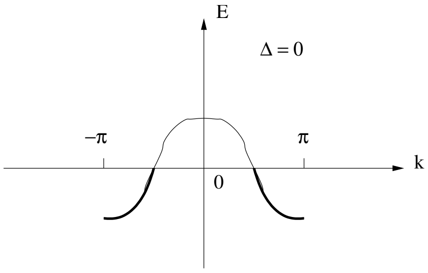

Figure 1: Excitation spectrum for an XY chain. At half filling (indicated as bold curve) the average polarisation $\langle S_z \rangle = 0$.

- The ground state energy $E = 2JN \displaystyle\int_{\pi/2}^{\pi} (dk)\, \cos(k) = -\frac{NJ}{\pi}$.
- It's also possible to calculate the correlation functions:

$$\begin{aligned} C(r) = \langle 0 | S_0^x\, S_r^x | 0 \rangle &= \frac{1}{4} \langle (S_0^+ + S_0^-)(S_r^+ + S_r^-) \rangle_0 \\ &= \frac{1}{4} \langle S_0^-\, S_r^+ \rangle_0 + \text{ c.c.} \\ &= \frac{1}{4} \left\langle 0 \left| \psi_0 \psi_r^\dagger \exp\left[i\pi \sum_{i=0}^{r} n_i \right] \right| 0 \right\rangle + \text{ c.c..} \end{aligned}$$

Replace the phase factor by its mean value, $\langle n_i \rangle = 1/2$:

$$\begin{aligned} C(r) &= \frac{1}{4} \left\langle 0 \left| \psi_0 \psi_r^\dagger \right| 0 \right\rangle \cdot e^{i\pi r/2} + \text{ c.c.} \\ &= \frac{1}{4N} \sum_k \left\langle 0 \left| e^{ikr} \psi(k) \psi^\dagger(k) \right| 0 \right\rangle \cdot e^{i\pi r/2} + \text{ c.c.} \\ &= \frac{1}{4} \int_{-\pi}^{\pi} (dk)\, e^{ikr} \cdot e^{i\pi r/2} + \text{ c.c.} \\ &= \frac{1}{4\pi r} \sin\left(\frac{\pi r}{2}\right) \cdot e^{i\pi r/2} + \text{ c.c.} = \frac{1}{8\pi r} \sin(\pi r). \end{aligned}$$

$\Rightarrow$ Oscillatory, i.e., antiferromagnetic correlation. Note the slow power-law decay of $C(r)$. So the 1D XY chain at $T = 0$ is not quite ordered (for $C(r)$ decays), nor is it fully disordered (for then $C(r) \sim \exp(-r/\xi) \sin(\pi r)$).

(b) Consider $\Delta = -1$ or less: Ferromagnetic Heisenberg chain

$$H = J \sum_k (\cos(k) - \Delta) \psi^\dagger(k) \psi(k) + \frac{J\Delta}{N} \sum_{p,q,k} e^{iq} \psi^\dagger(p+q) \psi^\dagger(k-q) \psi(k) \psi(p) + \frac{JN\Delta}{4}.$$

$\triangleright$ If $\Delta \leq -1$, then the 'kinetic' term is always positive $\Rightarrow$ we want an empty band as the ground state. An empty band also diagonalises the second term in the Hamiltonian.

$\Rightarrow$ Ground state is fully polarized. Excitations are spin waves.

(c) Case of arbitrary $\Delta > -1$: need to understand interacting fermions in 1D, i.e., a Luttinger Liquid. To learn more about this case, refer to the book by Gogolin, Nersesyan, and Tsvelik.