Appendix C

Continuous Spin Systems Near Two-Dimensions

The Hamiltonian of an n-component spin ferromagnet on a d-dimensional cubic lattice is given by

$$\beta H = -K \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j = \frac{K}{2} \sum_{\langle ij \rangle} \left[(\mathbf{S}_i - \mathbf{S}_j)^2 - 2 \right], \tag{C.1}$$

where $\mathbf{S}_i = (S_{i1}, \cdots S_{in})$ and $|S_i| = 1$. Dropping a constant contribution, the continuum limit of the latter generates the O(n) *non-linear σ-model*

$$\beta H = \frac{K}{2} \int d^d\mathbf{x}\, (\nabla \mathbf{S})^2, \tag{C.2}$$

where $|\mathbf{S}(\mathbf{x})| = 1$.

In the vicinity of the transition temperature, it is convenient to expand the spin degrees of freedom around the (arbitrary) direction of spontaneous symmetry breaking, $\mathbf{S}_0(\mathbf{x}) = (0, \cdots 0, 1)$,

$$\mathbf{S}(\mathbf{x}) = (\Pi_1(\mathbf{x}), \cdots \Pi_{n-1}(\mathbf{x}), \sigma(\mathbf{x})) \equiv (\mathbf{\Pi}(\mathbf{x}), \sigma(\mathbf{x})), \tag{C.3}$$

where $\sigma(\mathbf{x}) = (1 - \mathbf{\Pi}^2)^{1/2}$. Substituting this expression, and expanding σ we obtain

$$\begin{aligned} \beta H &= \frac{K}{2} \int d^d\mathbf{x} \left[(\nabla \mathbf{\Pi})^2 + (\nabla \sigma)^2 \right], \\ &= \frac{K}{2} \int d^d\mathbf{x} \left[(\nabla \mathbf{\Pi})^2 + \frac{1}{2} \left(\nabla \mathbf{\Pi}^2 \right)^2 + \cdots \right]. \end{aligned} \tag{C.4}$$

Treating the expansion to quadratic order, an estimate of the typical fluctuations gives

$$\begin{aligned} \langle |\mathbf{\Pi}(0)|^2 \rangle &= \int \frac{d^d\mathbf{q}}{(2\pi)^d} \langle |\mathbf{\Pi}(\mathbf{q})|^2 \rangle = \int_{1/L}^{1/a} \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{n-1}{K\mathbf{q}^2} \\ &= \frac{n-1}{K} \frac{S_d}{(2\pi)^d} \frac{a^{2-d} - L^{2-d}}{d-2} \overset{L\to\infty}{\sim} \begin{cases} T & d > 2, \\ \infty & d \leq 2. \end{cases} \end{aligned} \tag{C.5}$$

This confirms that the lower critical dimension is $d_c^l = 2$. Close to two-dimensions it is possible to study the critical behaviour in the vicinity of the transition using a perturbative renormalisation group.

C.1 Polyakov's Perturbative Renormalisation Group

The partition function of the non-linear σ-model can be written as

$$\begin{aligned} \mathcal{Z} &= \int D\mathbf{S}(\mathbf{x})\delta\left(\mathbf{S}^2(\mathbf{x})-1\right)e^{-\beta H}, \\ &= \int D\mathbf{\Pi}(\mathbf{x})D\sigma(\mathbf{x})\delta\left(\mathbf{\Pi}^2+\sigma^2(\mathbf{x})-1\right)e^{-\beta H}. \end{aligned} \tag{C.6}$$

The δ-function can be expanded as

$$\begin{aligned} &\int d\mathbf{\Pi}_i d\sigma(\mathbf{x})\delta\left(\mathbf{\Pi}_i^2+\sigma_i^2-1\right) \\ &= \int d\mathbf{\Pi}_i d\sigma(\mathbf{x})\delta\left(\sigma_i-(1-\mathbf{\Pi}_i^2)^{1/2}\right)\delta\left(\sigma_i+(1-\mathbf{\Pi}_i^2)^{1/2}\right) \\ &= \int\frac{d\mathbf{\Pi}_i}{2(1-\mathbf{\Pi}_i)^{1/2}}=\int\frac{d\mathbf{\Pi}_i}{2}e^{-(1/2)\ln(1-\mathbf{\Pi}_i^2)}. \end{aligned} \tag{C.7}$$

As a result, we obtain

$$\begin{aligned} \mathcal{Z} = \int D\mathbf{\Pi}(\mathbf{x})\exp\left[-\frac{\rho}{2}\int d^d\mathbf{x}\ln(1-\mathbf{\Pi}^2)\right] \\ \times\ \exp\left\{-\frac{K}{2}\int d^d\mathbf{x}\left[(\nabla\mathbf{\Pi})^2+\left(\nabla(1-\mathbf{\Pi}^2)^{1/2}\right)^2\right]\right\}, \end{aligned} \tag{C.8}$$

where ρ denotes the density of states.

Expanding the Hamiltonian perturbatively in Π, we find that $K\langle\mathbf{\Pi}^2\rangle\sim O(1)$, $K(\nabla\mathbf{\Pi}^2)^2\sim O(K^{-1})$, and $\rho\mathbf{\Pi}^2\sim O(K^{-1})$. This suggests that we define

$$\beta H_0=\frac{K}{2}\int d^d\mathbf{x}\left(\nabla\mathbf{\Pi}\right)^2, \tag{C.9}$$

as the unperturbed Hamiltonian and treat

$$U=\frac{K}{2}\int d^d\mathbf{x}\left(\mathbf{\Pi}\cdot\nabla\mathbf{\Pi}\right)^2-\frac{\rho}{2}\int d^d\mathbf{x}\mathbf{\Pi}^2. \tag{C.10}$$

The latter can be expanded in terms of Fourier modes as

$$\begin{aligned} U &= -\frac{K}{2}\int\frac{d^d\mathbf{q}_1d^d\mathbf{q}_2d^d\mathbf{q}_3}{(2\pi)^{3d}}\mathbf{q}_1\cdot\mathbf{q}_3\Pi_\alpha(\mathbf{q}_1)\Pi_\alpha(\mathbf{q}_2)\Pi_\beta(\mathbf{q}_3)\Pi_\beta(-\mathbf{q}_1-\mathbf{q}_2-\mathbf{q}_3) \\ &\quad -\frac{\rho}{2}\int\frac{d^d\mathbf{q}}{(2\pi)^d}|\mathbf{\Pi}(\mathbf{q})|^2, \end{aligned} \tag{C.11}$$

where

$$\langle\Pi_\alpha(\mathbf{q}_1)\Pi_\beta(\mathbf{q}_2)\rangle_0=\frac{1}{K\mathbf{q}_1^2}\delta_{\alpha\beta}(2\pi)^d\delta^d(\mathbf{q}_1+\mathbf{q}_2). \tag{C.12}$$

The perturbation can be represented diagrammatically as in fig. C.1a where the slashes denote those fields that carry the momenta.

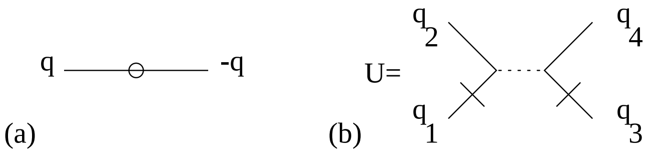

Figure C.1: Diagrammatic representation of vertices.

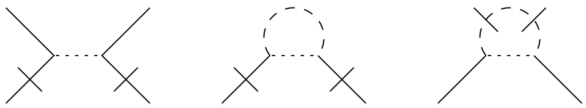

Figure C.2: Diagrams contributing to the first order of perturbation theory.

C.1.1 Perturbative Renormalisation Group

To apply the perturbative RG, we separate the fields into components,

$$\mathbf{\Pi}(\mathbf{q}) = \begin{cases} \mathbf{\Pi}_<(\mathbf{q}) & 0 < |\mathbf{q}| < \Lambda/b, \\ \mathbf{\Pi}_>(\mathbf{q}) & \Lambda/b < |\mathbf{q}| < \Lambda. \end{cases} \tag{C.13}$$

The partition function can then be expressed as

$$\begin{aligned} \mathcal{Z} &\sim \int D\mathbf{\Pi}_< D\mathbf{\Pi}_> e^{-\beta H_0[\mathbf{\Pi}_<]-\beta H_0[\mathbf{\Pi}_>]-U[\mathbf{\Pi}_<,\mathbf{\Pi}_>]}, \\ &\sim \int D\mathbf{\Pi}_< e^{-\delta f_b^0-\beta H_0[\mathbf{\Pi}_<]-\ln\left\langle e^{-U[\mathbf{\Pi}_<,\mathbf{\Pi}_>]}\right\rangle_0^>}, \end{aligned} \tag{C.14}$$

where δf_b^0 is a constant arising from the bare integral. The renormalisation of the Hamiltonian can be obtained from the cummulant expansion

$$-\ln\left\langle e^{-U}\right\rangle = \langle U\rangle - \frac{1}{2}\left\langle U^2\right\rangle_0^c + O(U^3). \tag{C.15}$$

Treating the expansion to first order (fig. C.2), only two terms give a non-trivial contribution. The others vanish or give a constant. As a result we obtain for each diagram

$$D_1 = -\frac{K}{2}\int_0^{\Lambda/b}\frac{d^d\mathbf{q}}{(2\pi)^d}\mathbf{q}^2|\mathbf{\Pi}_<|^2(\mathbf{q})\frac{I_d(b)}{K}, \tag{C.16}$$

where

$$I_d(b) = \int_{\Lambda/b}^{\Lambda}\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{1}{\mathbf{k}^2} = K_d\int_{\Lambda/b}^{\Lambda} dk k^{d-3}. \tag{C.17}$$

Similarly, using the formula for the density of states

$$\rho = \frac{N}{V} = \int_0^{\Lambda}\frac{d^d\mathbf{q}}{(2\pi)^d} = b^d\int_0^{\Lambda/b}\frac{d^d\mathbf{q}}{(2\pi)^d}, \tag{C.18}$$

we obtain

$$D_2 = -\frac{1}{2}\int_0^{\Lambda/b}\frac{d^d\mathbf{q}}{(2\pi)^d}|\mathbf{\Pi}_<(\mathbf{q})|^2\rho(1-b^{-d}). \tag{C.19}$$

Taking these contributions we obtained the renormalised Hamiltonian

$$\begin{aligned}-\widetilde{\beta H} &= \delta f_b^0 + \delta f_b^1 - \frac{\widetilde{K}}{2}\int_0^{\Lambda/b} d^d\mathbf{x}(\nabla\mathbf{\Pi}_<)^2 + \frac{\rho}{2}b^{-d}\int_0^{\Lambda/b} d^d\mathbf{x}|\mathbf{\Pi}_<|^2\\ &\quad -\frac{K}{2}\int_0^{\Lambda/b} d^d\mathbf{x}\left(\Pi_{<\alpha}\nabla\Pi_{<\alpha}\right)^2 + O(K^{-2}),\end{aligned} \tag{C.20}$$

where

$$\widetilde{K} = K\left(1+\frac{I_d(b)}{K}\right). \tag{C.21}$$

Applying the rescaling $x' = x/b$ and renormalising the spins,

$$\mathbf{S}' = \frac{\mathbf{S}}{\xi}, \qquad \Pi_< = \xi\Pi', \tag{C.22}$$

we obtain (up to a constant)

$$\begin{aligned}-\beta H' &= -\frac{\widetilde{K}}{2}b^{d-2}\xi^2\int_0^{\Lambda} d^d\mathbf{x}'(\nabla\mathbf{\Pi}')^2 + \frac{\rho}{2}\xi^2\int_0^{\Lambda} d^d\mathbf{x}'|\mathbf{\Pi}'|^2\\ &\quad -\frac{K}{2}b^{d-2}\xi^4\int_0^{\Lambda} d^d\mathbf{x}'(\Pi'_\alpha\nabla'\Pi'_\alpha)^2 + O(K^{-2}),\end{aligned} \tag{C.23}$$

where

$$K' = b^{d-2}\xi^2 K\left(1+\frac{I_d(b}{K}\right) \tag{C.24}$$

Finding ξ

$$\mathbf{S} = \left(\Pi_{<1}+\Pi_{>1},\cdots\left(1-\mathbf{\Pi}_<^2-\mathbf{\Pi}_>^2\right)^{1/2}\right) \tag{C.25}$$

$$\begin{aligned}\widetilde{\mathbf{S}} = \langle\mathbf{S}\rangle_0^> &= \left(\Pi_{<1},\cdots,1-\mathbf{\Pi}_<^2/2-\left\langle\mathbf{\Pi}_>^2\right\rangle/2\right)\\ &= \left(1-\left\langle\mathbf{\Pi}_>^2\right\rangle/2\right)\left(\Pi_{<1},\cdots,1-\mathbf{\Pi}_<^2/2\right) = \xi\mathbf{S}'\end{aligned} \tag{C.26}$$

Thus we obtain

$$\xi = 1-\frac{1}{2}(n-1)\int_{\Lambda/b}^{\Lambda}\frac{d^d\mathbf{k}}{(2\pi)^d}\frac{1}{K\mathbf{k}^2} = 1-\frac{n-1}{2K}I_d(b), \tag{C.27}$$

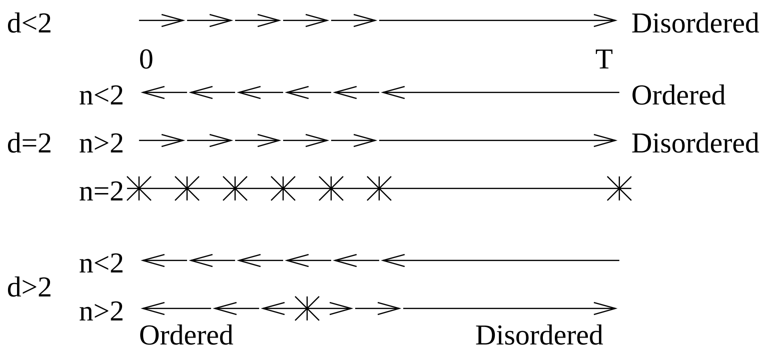

Figure C.3: RG flow predicted from the first order of perturbation theory.

and

$$K' = Kb^{d-2}\left[1 - \frac{n-2}{K} I_d(b)\right]. \tag{C.28}$$

Setting $b = e^{\delta\ell} = 1 + \delta\ell$, we obtain the differential recursion relations

$$\begin{aligned} K + \delta\ell \frac{dK}{d\ell} &= K\left(1 + (d-2)\delta\ell\right)\left(1 - \frac{n-2}{K} K_d \Lambda^{d-2}\delta\ell\right), \\ \frac{dK}{d\ell} &= (d-2)K - (n-2)K_d\Lambda^{d-2}. \end{aligned} \tag{C.29}$$

Assigning the temperature $T = K^{-1}$, we obtain

$$\frac{dT}{d\ell} = -\frac{1}{K^2}\frac{dK}{d\ell} = -(d-2)T + (n-2)K_d\Lambda^{d-2}T^2 + O(T^3). \tag{C.30}$$

The corresponding RG flows are shown in fig. C.3. Solving this equation for the fixed point $dT/d\ell = 0$, we obtain

$$T^* = \frac{d-2}{(n-2)K_d\Lambda^{d-2}} = \frac{2\pi\epsilon}{(n-2)} + O(\epsilon^2), \tag{C.31}$$

where $d = 2 + \epsilon$.

Linearising the RG flow in the vicinity of the fixed point we obtain

$$\frac{d\delta T}{d\ell} = (-\epsilon + 2(n-2)K_d\Lambda^{d-2}T^*)\delta T = \epsilon\delta T. \tag{C.32}$$

Integrating, we obtain $\delta T(\epsilon) = b^\epsilon \delta T(0)$. From this we obtain $y_t = \epsilon + O(\epsilon^2)$ from which we deduce that the thermal exponents are independent of n at this order. In $d = 3$, with $n = 3$ this predicts

$$\nu = \frac{1}{y_t} = \frac{1}{\epsilon} = 1. \tag{C.33}$$

This compares with the best estimate from numerics of $\nu = 0.7$ (and an estimate of $\nu = 0.67$ from the $4 - \epsilon$ expansion).

To obtain the magnetic exponent, it is necessary to add a term

$$-\int d^d\mathbf{x}\mathbf{h}\cdot\mathbf{S}, \tag{C.34}$$

which under RG becomes

$$-b^d\xi\int d^d\mathbf{x}\mathbf{h}\cdot\mathbf{S}' \tag{C.35}$$

Thus, using $h' = b^d\xi h = b^{y_h}h$, we find

$$b^{y_h} = b^d\left[1 - \frac{n-1}{2K^*}I_d(b)\right], \tag{C.36}$$

This implies that

$$\frac{dh}{d\ell} = h\left[d - \frac{n-1}{2(n-2)}\epsilon\right] \tag{C.37}$$

from which we obtain

$$y_h = 2 + \frac{n-3}{2(n-2)}\epsilon. \tag{C.38}$$

Using the exponent identities, we obtain

$$\gamma = \nu(2y_h - d) = \frac{1}{\epsilon}\left(2 - \frac{\epsilon}{n-2}\right). \tag{C.39}$$

This predicts a value $\gamma = 1$ for $d = 3$ and $n = 3$ which compares with the best estimate of 1.38. Finally, using the relation $\gamma = \nu(2-\eta)$ we obtain $\eta = \epsilon/(n-2)$.